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Analogical Reasoning with Rational Numbers: Semantic Alignment Based on Discrete Versus Continuous Quantities

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Abstract

Non-integer rational numbers, such as fractions and decimals, pose challenges for learners, both in conceptual understanding and in performing mathematical operations. Previous studies have focused on tasks involving access and comparison of integrated magnitude representations, showing that adults have less precise magnitude representations for fractions than decimals. Here we show the relative effectiveness of fractions over decimals in reasoning about relations between quantities. We constructed analogical reasoning problems that required mapping rational numbers (fractions or decimals) onto pictures depicting either part-whole or ratio relations between two quantities. We also varied the ontological nature of the depicted quantities, which could be discrete, continuous, or continuous but parsed into discrete components. Fractions were more effective than decimals for reasoning about discrete and continuous-parsed (i.e., discretized) quantities, whereas neither number type was particularly effective in reasoning about continuous quantities. Our findings show that, when numbers serve as models of quantitative relations, the ease of relational mapping depends on the analogical correspondence between the format of rational numbers and the quantity it models.

Keywords: analogy; relational reasoning; number concepts; fractions; decimals; semantic alignment; math education

Introduction

Mathematical Understanding as Relational Reasoning

Mathematics is in essence a system of relations among concepts based on quantities. A core problem with math education, particularly in the United States, is that greater focus is placed on execution of mathematical procedures than on understanding of quantitative relations (Richland, Stigler & Holyoak, 2012; Stigler & Hiebert, 1999; Rittle-Johnson & Star, 2007). An early manifestation of this problem involves teaching of non-integer rational numbers in the standard curriculum—typically, first fractions and subsequently decimals. Students often leave middle-school (and often enter community college: see Stigler, Givvin & Thompson, 2010; Givvin, Stigler & Thompson, 2011) without having grasped how fractions relate to decimals, or how either number type relates to integers. This conceptual disconnection in turn contributes to a compartmentalization of mathematical operations (e.g., multiplication of fractions is treated as unrelated to multiplication of integers; Siegler et al., 2011; Siegler & Pyke, 2012).

Although mathematical relations are typically construed as internal to the formal system of mathematics, the application of mathematics to real-world problems also depends on grasping relations between mathematical concepts and the basic ontological distinctions among the concepts to which mathematics must be applied. Rather than treating mathematical concepts as purely formal, both children and adults are naturally guided by a process of *semantic alignment*, which favors mapping certain mathematical concepts (and their associated operations) onto certain conceptual types. Bassok, Chase and Martin (1998) demonstrated that the basic mathematical operations of addition, subtraction, multiplication, and division are typically conceptualized within a system of relations between mathematical values and objects in the real world. Specific mathematical operators are semantically aligned with particular relationships among real-world objects. For example, addition is aligned with categorical object relations (e.g., people find it natural to add two apples plus three oranges, because both are subtypes of a common category, fruit), whereas division is aligned with functional object relations (e.g., a natural problem would be to divide ten apples between two baskets). Semantic alignment has been demonstrated with both children and adults (e.g., Martin & Bassok, 2005), and for many adults the process is highly automatic (Bassok, Pedigo, & Oskarsson, 2008). Although natural semantic alignments are implicitly acknowledged in the construction of textbook word problems (Bassok et al., 1998), teachers seldom discuss these alignments with their students. This gap may contribute to the difficulty of conveying how and why mathematical formalisms "matter" in dealing with real-world problems.

Discreteness Versus Continuity

A particularly important ontological distinction relevant to mathematical modeling involves the nature of quantities, which can be viewed as either *discrete* or *continuous*. Roughly, some entities are viewed as comprising a set of individual objects (e.g., a number of apples in a basket), whereas others are viewed as a continuous mass without individuation (e.g., a bucket of water). Although continuous, as well as, discrete quantities can be subdivided, in the case of continuous quantities the divisions are arbitrary in the sense that they do not isolate conceptual parts (e.g., one could distinguish between subsets of red and green apples in a basket by saying that 2/3 of the apples are red and 1/3 are green, but there is no psychological difference between the water contained in 2/3 of a bucket and in the complementary 1/3 of the bucket). Importantly, discreteness versus continuity is a distinction based fundamentally not on physics, but on psychology. For example, a pile of sand is viewed as a continuous quantity even though we know it is composed of individual grains, because those units are too small and interchangeable to be typically viewed as "important". The impact of this basic ontological distinction has been documented both in young babies (e.g., Spelke, Breilinger, Macomber, & Jacobson, 1992) and in adults (Bassok & Olseth, 1995). For example, Bassok and Olseth found that college students viewed an increase in attendance at an annual conference as discrete (since it is based on a change between magnitudes associated with two discrete events well-separated in time), but viewed an annual increase in a city's population growth as continuous (since it is based on changes stemming from the psychologically-constant process of gaining and losing undifferentiated individual residents).

The ontological distinction between discreteness and continuity underlies the linguistic distinction between count and mass nouns, which is syntactically important in English and many other natural languages (Bloom & Wynn, 1997). Infants and young children are able to make distinctions among continuous quantities (Clearfield & Mix, 2001; Fiegenson, Carey & Spelke, 2002). However school-aged children have an advantage when performing operations with discrete quantities (e.g., counting; Gelman, 1993) over performing operations with their continuous counterparts (e.g., measurement in general; Nunes, Light & Mason, 1993). Indeed, measurement of continuous quantities depends on the introduction of arbitrary equal-sized units, which serve to parse a continuous whole into countable subparts (e.g., a continuous length can be broken down into some number of inches or centimeters). The ability to discretize continuous concepts (as contrasted with the lack of a natural operation to make discrete concepts continuous) leads to asymmetries in transfer of mathematical operations. For example, college students can transfer the equation for calculating the sum of an arithmetic progression (discrete concept) to solve a physics problem requiring solving for final velocity after constant acceleration (continuous concept), but find transfer in the opposite direction (continuous to discrete) nearly impossible (Bassok & Holyoak, 1989; Bassok & Olseth, 1995).

Fractions as Relational Representations

As the first non-integer number type introduced to elementary-school students, fractions pose particular challenges. Research indicates that children have difficulty integrating fractions into their already well-established understanding of whole numbers (Staflyidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010; Ni & Zhou, 2005), and even adults at community colleges seem to lack fundamental understanding of how to use fractions (Stigler et al., 2010). Research on understanding fractions has primarily focused on the ability to grasp and manipulate their integrated magnitude value associated with the *a/b* form. Although adults can compare fractions based on integrated magnitudes (Schneider & Siegler, 2010), this process is very slow and error-prone relative to performing the same task with decimal equivalents (DeWolf, Grounds, Bassok & Holyoak, in press). The difficulty of making magnitude comparisons with fractions presumably reflects their bipartite structure (numerator divided by denominator), which makes them both formally and conceptually distinct from integers. In contrast, decimals have a unitary structure more similar to integers (though not identical; Cohen, 2010).

However, even though the internal structure of fractions apparently hinders access to precise integrated magnitudes, this same structure may facilitate understanding of key relations. In particular, the *a/b* form can be aligned with the concepts underlying relations such as part/whole, subset/set, ratio, and rate. When children are first taught the concept of a fraction, some type of pictorial representation is often provided, such that each of the two values in the fraction are structurally aligned with two separate elements in the picture. For example, take the very common example of cutting up a pizza pie into pieces. A child might be taught that 4/5 is equivalent to 4 slices of a pizza pie that is divided into 5 slices. This type of mapping is also encouraged with verbal examples (e.g., 4 out of every 5 dentists recommend a certain toothpaste). Such instructional practices highlight the relational nature of fractions and encourage children to reason about fractions relationally.

We propose that semantic alignment will also modulate people's understanding of fractions. Fractions seem particularly appropriate as models of relations between sets of discrete elements. The representations typically used to teach fractions focus on discrete countable units that can map to the numerator and denominator values. For example, the pizza is sliced into exactly the number of pieces in the denominator before the numerator pieces are counted up. Rapp and Bassok (in preparation) reviewed a math textbook series and found that very rarely are students encouraged to think about fractions with continuous representations, such as a number line. In fact, continuous measures (e.g., length, weight) are almost exclusively represented with decimals. Rapp and Bassok also found that, consistent with this

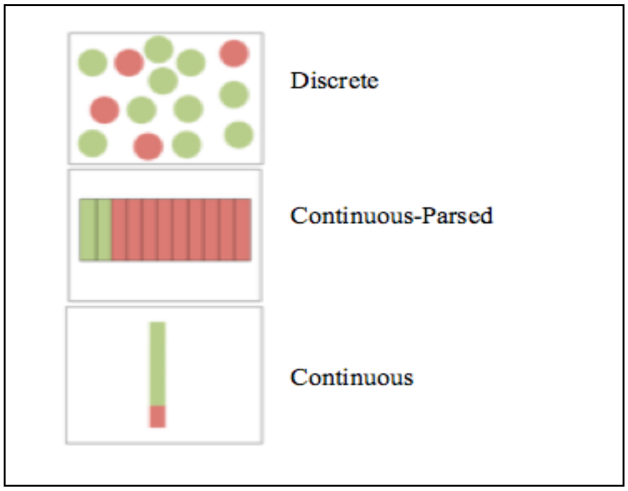

Figure 1: Examples of types of pictures used in analogy problems.

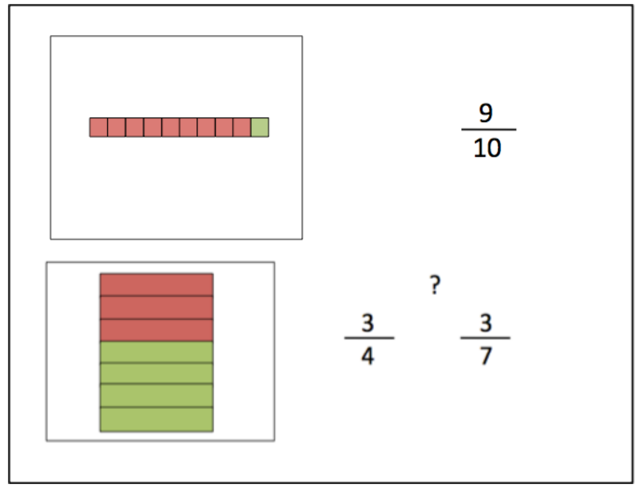

Figure 2: Example of an analogy problem (part/whole fraction trial with continuous-parsed pictures).

training, college students show a preference for using fractions rather than decimals to describe relations between discrete quantities, and use decimals to describe magnitudes of continuous quantities.

Analogical Reasoning with Quantitative Relations

Our study was designed to test the hypothesis that, due to their relational structure (*a/b*), fractions are better suited than decimals for representing relations between countable quantities. To this end, we compared analogical reasoning with either fractions or decimals, while varying the ontological distinction between discrete and continuous concepts. Figure 1 shows examples of variations in discreteness versus continuity. The pictorial stimuli were based on discrete elements (top), continuous rectangles (bottom), or continuous rectangles parsed into discrete units (middle). We hypothesized that semantic alignment would yield higher accuracy and faster response times for solving analogies using fractions rather than decimals for the discrete and continuous-parsed pictures. The fraction advantage was predicted to disappear or even reverse for the continuous pictures, where the semantic alignment is most difficult.

Method

Participants

Participants were 52 undergraduates at the University of California, Los Angeles (mean age = 21; 30 females) who received course credit, randomly assigned in equal numbers to the two between-subjects conditions.

Materials and Design

The study was a 2 (number type: fractions vs. decimals) X 2 (relation type: ratios vs. part/whole fractions) X 3 (picture type: continuous, continuous-parsed, discrete) design, with number type as a between-subjects factor and relation type and picture type as within-subjects factors.

The analogy problems were constructed using each of the three ontological types illustrated in Figure 1: discrete, continuous-parsed, and continuous. An example problem appears in Figure 2. These analogy problems were in the format A:B :: C:D vs. D', where the source analog (A:B) consisted of a picture and a number (fraction or decimal). The task required making a choice of the correct number to complete the target analog. The number type was always the same across the source and target.

Solving an analogy problem required first identifying the relationship in the A picture characterized by the number given as B. This relationship could be part-whole or a ratio between two parts. In Figure 2, the A picture indicates 4 green units out of a total of 6, making a part-whole relation of 4/6 (.67 in a matched problem using decimals). An alternative ratio relation in Figure 2 is based on the units of red relative to green (i.e., 2/4, or .50 in decimal notation). Once the higher-order relation between A and B was extracted, the solution required identifying the same relation type in target picture C, and choosing the corresponding number as D term. D' mapped to the alternative relationship.

As Figure 2 illustrates, the same two colors were used in the A and C pictures, and the color relationship was maintained, such that the same color mapped to the same part of the relation in both A and C. This constraint served to identify which part (lesser or greater) mapped to the numerator in a ratio relation. Color assignments varied across trials, so the same color might indicate the lesser subset on one trial and the greater subset on another. The actual test trials contained only red and green colors (practice trials were given that had yellow and brown colors). The discrete items were circles, squares, crosses, trapezoids, and cloud-like shapes. Continuous and continuous-parsed items differed in width, height and

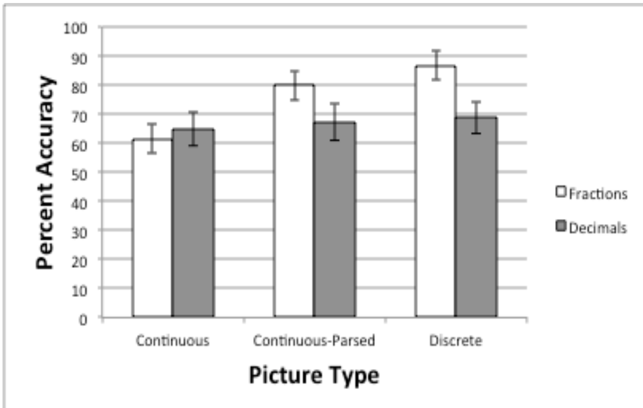

Figure 3: Percent accuracy for solving analogy problems using fractions or decimals for each quantity type.

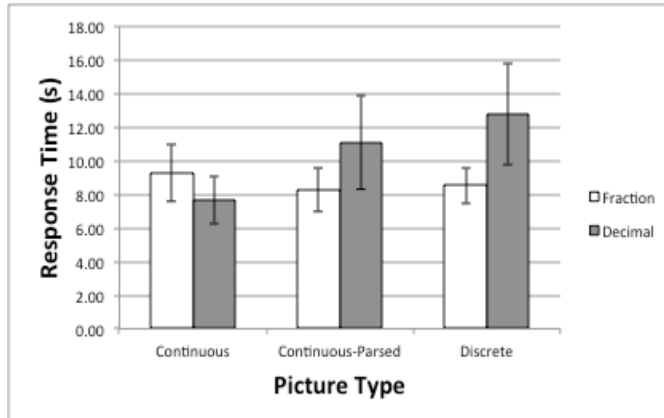

Figure 4: Response times for correctly solving analogy problems using fractions or decimals for each quantity type.

orientation (vertical or horizontal). For each trial, the source and target were randomly assigned for each participant so that the only thing that was consistent between the two was the higher-order relationship (ratio or part/whole) and the color mapping. The fractions and decimals were always less than one and decimals were shown rounded to two decimal places.

Procedure

Stimuli were displayed with Macintosh computers using Superlab 4.5, and response times and accuracy were recorded. Extensive instructions and practice was provided prior to beginning the test trials. Participants were told that there were two different types of relations between the pictures and values. For the ratio relationship, participants were shown a picture with 1 O and 2 X's. For the fractions condition this was explained as "1/2 amount of O's per amount of X's;" for the decimals condition it was explained as ".50 amount of O's per amount of X's." The part/whole relationship was represented with a picture of 2 O's and 3 X's. For the fractions condition this was explained as "2/5 of the total is the amount of O's;" for the decimal condition it was explained as ".40 of the total is the amount of O's." The first of these explanations of the ratio and part/whole relations was shown with discrete items. The following screen showed the same values paired with continuous-parsed pictures. A third screen showed the same values paired with continuous pictures.

After this introduction, participants completed an example trial in which they were shown the source (A:B), asked to figure out the type of relation (ratio or part/whole) in their head, and press the space bar. After the space bar was pressed, the target (C:D vs. D') was shown on the screen below the source so that the two components were on the screen simultaneously. Participants were asked to select which of two numbers (D or D') shared the same relationship with the picture as the relationship provided in the source. Half of the time, D appeared on the right side of the screen. They made their selection by pressing the *z* key for the number shown on the left and the *m* key for the number shown on the right. The *z* and *m* keys were labeled with "L" and "R", respectively, so that participants could remember which key went with each number. After completing the initial example trial, participants were shown the correct answer, with an explanation of which relationship was shared between the source and target.

Participants then completed 12 practice trials. Feedback was given for incorrect trials, in the form of a red "X" on the screen. After the practice trials had been completed, a screen was displayed informing participants that the actual test trials were beginning. For each trial, the source was shown, then the participant pressed the spacebar when they determined the relationship. The target was then shown in addition to the source. Feedback was continued for incorrect trials. There were 72 test trials (12 for each of the 6 within-subjects conditions). The specific pictures, numbers, and pairings used in the test trials were different from those used in practice trials. Relation types and picture types were shown in a different random order for every participant.

Results

Accuracy and mean response time (RT) on correct trials were computed for each condition for each participant. A mixed factors ANOVA was used to compare differences in RT and accuracy. No reliable overall differences were obtained between the two relation types (part-whole and ratio) on either measure; hence all results are reported after collapsing across this variable. Figure 3 presents the pattern of performance based on accuracy, and Figure 4 presents the pattern based on mean correct RT. Both dependent measures revealed an overall advantage for solving analogies based on fractions rather than decimals, with the advantage most pronounced for pictures of discrete quantities. For accuracy, both number type, $F(1, 50) = 8.65$, $p = .005$), and picture type, $F(2, 49) = 33.52$, $p < .001$, were highly reliable, as was the interaction of the two factors, $F(2, 49) = 25.20$, $p < .001$. Planned comparisons indicated that accuracy was higher for fractions than decimals for the discrete condition (87% vs.

66%; $t(50) = 5.38$, $p < .001$) and the continuous-parsed condition (80% vs. 67%; $t(50) = 3.17$, $p = .003$), but did not differ for the continuous condition (61% vs. 65%; $t(50) = .93$, $p = .36$).

RTs were measured from the onset of the source display on the screen to the selection of the target answer. Response times for incorrect answers were excluded from analyses. In addition, outliers were trimmed to exclude any times that were greater than three standard deviations from the mean (roughly 2% of response times). As shown in Figure 4, the RT pattern closely resembled that for accuracy. In particular, there was a reliable interaction between number type and picture type, $F(2, 49) = 16.19$, $p < .001$. Planned comparisons indicated that RTs were faster with fractions than decimals for the discrete condition (8.5 s vs. 12.8 s; $t(50) = 2.70$, $p = .01$), with a strong trend for the continuous-parsed condition (8.3 s vs. 11.2 s; $t(50) = 1.87$, $p = .067$). RTs for fractions versus decimals did not differ reliably for the continuous condition (9.3 s vs. 7.7 s; $t(50) = 1.45$, $p > .15$).

Discussion

The results of the current study demonstrated an overall advantage for fractions over decimals in a relational task. Moreover, this advantage was moderated by the ontological nature of the depicted quantities. Participants were better able to extract relationships for discrete and continuous-parsed pictures when fractions were mapped to the quantities, rather than decimals. There was no difference in performance on the continuous pictures between fractions and decimals. This pattern suggests that fractions are semantically-aligned with relations between countable, discrete quantities. Performance with decimals was relatively flat (and generally poorer) for all picture types.

These results support two basic claims about semantic alignment for specific types of quantities. First, people can and do align quantities with numbers. Second, ease of alignment depends on two factors: the type of number format (fractions vs. decimals), and the type of quantities (countable, i.e., discrete and continuous-parsed, vs. continuous).

The central difference between fractions and decimals is that their formats provide an explicit representation of relations (fractions) or of relation magnitudes (decimals). That is, fractions have a bipartite structure (*a/b*) that expresses a specific relationship between two natural numbers, *a* and *b*. Decimals represent the magnitudes of such fractional relations. This difference has important implications for how people align these numbers with specific quantities. For fractions, alignment should be simple when the numerator and denominator can be readily mapped onto distinct subsets, A and B. Our results show that this is indeed the case when A and B are comprised of countable entities, depicted by the discrete and continuous-parsed picture types. However, alignment should be difficult when A and B are continuous quantities, as the task becomes more like a magnitude assessment. Despite the explicit relation (*a/b)*, it is difficult to assess the magnitude of A and the magnitude of B, which makes the overall mapping more complicated. Decimals represent magnitudes of relations without specifying the relational parts. Hence, mapping to the A and B sets is difficult irrespective of whether the sets are shown as discrete or continuous quantities.

The current pattern of results is consistent with the schooling experience of our participants (Rapp & Bassok, in preparation). Typically, students learn about fractions from part/whole and set/subset examples (Sophian, 2007; Mack, 1993). However, the goal of such examples is not to help children understand that fractions represent relations. Rather, they are provided to help children understand the existence of values smaller than 1. That is, as discussed earlier, the main focus of initial instruction about fractions is to convey their magnitude. This focus is problematic because, while fractions are well-suited for representation of relations, they are poorly suited for representation of magnitudes (DeWolf et al., in press; Stigler et al., 2010). Our findings also suggest that if decimals were taught prior to fractions, children might have a better opportunity to learn about magnitudes smaller than 1. Because decimals have a unitized format, like whole numbers, they might provide an easier opportunity for children to master the idea of magnitudes smaller than 1. Fractions, then, might be taught later than decimals with an emphasis on their status as a relationship between two natural numbers. The magnitude of such relational representations would not be limited to values smaller than 1 (ratios).

Interestingly, Moss and Case (1999) implemented a curriculum with 4th graders in Canada that reorganized the order of rational number instruction. Children were first taught percentages (in the context of volumes and on number lines), then decimals, and lastly fractions. Fractions were explained simply as another way to represent a decimal. By contrast, typical curricula describe teaching decimals as another way to represent a fraction. Moss and Case found that children taught number types in this novel sequence suffered less interference from whole-number strategies when using other rational numbers, and achieved a deeper understanding of them. Though Moss and Case did not emphasize fractions in the relational context we have discussed here, it seems that introducing the idea of magnitudes less than 1 with decimals rather than fractions may be preferable.

In summary, understanding how non-integer rational numbers align to specific types of quantities, and how format can affect ease of semantic alignment, has important implications for how we conceptualize and teach fractions and decimals. It is important to foster understanding of fractions beyond simple algorithmic procedures, and to bolster conceptual understanding in order to address the difficulties children and adults face in understanding fractions.

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